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THE STABILITY OF THE STEADY MOTIONS OF A ROTOR SYSTEM WITH A FLUID USING A DISCRETE MODEL†

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A discrete model of a rotor system with a fluid is proposed. The model contains a rotating disc, which is directly and symmetrically seated on a spindle located in an isotropic, viscoelastic mounting, and a ring which slides with friction over the disc. There is a viscoelastic coupling between the centres of the disc and the ring. The disc simulates the rotor and the ring simulates the fluid mass of a filler. When the ring slides over the disc, an interactive force arises which is directed at an angle to the relative velocity. It is shown that, with a correct choice of the parameters, the model enables an approximate determination to be made of the domain of stability of steady rotation in the plane of the parameters of the viscoelastic mounting of the axis. It is established that, on leaving the domain of stability, Andropov–Hopf bifurcation occurs and a periodic motion of the type of a circular precession is generated (in a "soft" or "hard" way) from the state of steady rotation.

Mathematical models of rotor systems with a fluid which involve the Navier–Stokes equations are quite complicated for a stability analysis [1]. The stability of a rotor, partially filled with a fluid, was investigated in [2], subject to the condition that the angular velocity of rotation is constant, and the boundaries of domains with a different degree of instability were constructed in the parameter space of the problem. The difficulties of investigating distributed models have prompted an analysis of the possibility of constructing discrete models [3], which could describe some important features of the behaviour of a rotor system containing a fluid with satisfactory accuracy.

In actual systems, small changes in the parameters can occur over time and this evolution of the parameters can lead to a state when the system reaches the boundary of the stability domain. If, at the same time, a violation, which is as small as may be desired, of a "dangerous" boundary occurred, the system will pass into a new state which cannot be approximated to the initial state by choosing the violation of the boundary to be sufficiently small (for definitions of "safe" and "dangerous" boundaries, see [4, 5]). It would therefore seem desirable to supplement the investigation of stability with an investigation of the nature of the boundaries which, as is shown below, is fairly straightforward to do for a discrete model.

1. DISCRETE MODEL

A disc of mass m_d and radius R_d, located in a non-linear, viscoelastic mounting on a spindle, rotates at a constant angular velocity Ω in a horizontal plane (Fig. 1). A ring of mass m_r and radius R_r is attached to the centre of the disc. we will assume that the angular displacements of the axes of the disc and the ring are negligibly small and that all the points of the system can only move in the horizontal plane, that is, the plane perpendicular to the axis of steady rotation. We will now introduce two system of coordinates in the plane of motion of the disc and ring: a fixed system of coordinates Oxy, the origin of which is associated with the axis of the steady rotation, and a moving system of coordinates $O_1\xi\eta$ which coincides with the centre of the disc (Fig. 2). The centre of the ring is denoted by O_2.

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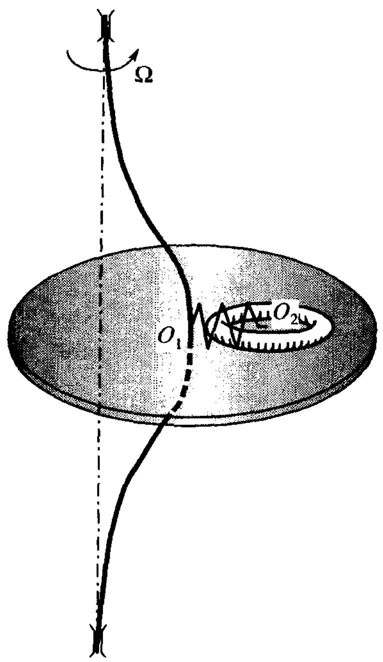

Fig. 1

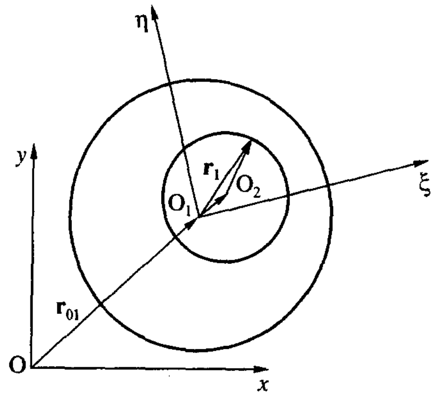

Fig. 2

The force acting on the disc from the side of the non-linear, viscoelastic mounting has the form

$$\mathbf{f}_d = -(k_d + k_{2d}|\mathbf{r}_{01}|^{\alpha})\mathbf{r}_{01} - (\eta_d + \eta_{2d}|\dot{\mathbf{r}}_{01}|^{\beta})\dot{\mathbf{r}}_{01}, \quad 0 < \alpha, \beta < 2 \tag{1.1}$$

where $\mathbf{r}_{01} = x_1\mathbf{e}_x = y_1\mathbf{e}_y$ is the radius vector from the origin of the fixed system of coordinates to the point O_1. Elastic and friction forces also act between the disc and the ring

$$\mathbf{f}_r = -k_r(\mathbf{r}_{02} - \mathbf{r}_{01}) - \eta_r\int(\mathbf{v}_r - \mathbf{v}_d)dl \tag{1.2}$$

where

$$\mathbf{v}_r - \mathbf{v}_d = (\dot{x}_2 - \dot{x}_1 - (y - y_2)\dot{\varphi}_2 + (y - y_1)\dot{\varphi}_1)\mathbf{e}_x + (\dot{y}_2 - \dot{y}_1 + (x - x_2)\dot{\varphi}_2 - (x - x_1)\dot{\varphi}_1)\mathbf{e}_y$$

$\mathbf{v}_d = \dot{\mathbf{r}}_{01} + [\boldsymbol{\omega}_1, \mathbf{r}_1]$ is the velocity of the point $M(x, y)$ of the disc relative to the fixed system of coordinates Oxy, $\mathbf{r}_1 = (x - x_1)\mathbf{e}_x + (y - y_1)\mathbf{e}_y$ is a radius vector from the point O_1 to the point M, $\boldsymbol{\omega}_1 = \dot{\varphi}_1\mathbf{e}_z$ is the angular velocity of rotation of the disc, $\mathbf{v}_r = \dot{\mathbf{r}}_{02} + [\boldsymbol{\omega}_2, \mathbf{r}_2]$ is the velocity of the point of the ring which is in contact with the point M of the disc, $\mathbf{r}_{02} = x_2\mathbf{e}_x + y_2\mathbf{e}_y$, $\mathbf{r}_2 = (x - x_2)\mathbf{e}_x + (y - y_2)\mathbf{e}_y$ is the radius vector from point O to the point O_2, and $\boldsymbol{\omega}_2 = \dot{\varphi}_2\mathbf{e}_z$ is the angular velocity of rotation of the ring.

The expressions for the potential and kinetic energy of the disc (subscript d) and the ring (subscript r) have the form

$$U_d = \frac{k_d}{2}(x_1^2 + y_1^2) + \frac{k_{2d}}{\alpha + 2}(x_1^2 + y_1^2)^{(\alpha + 2)/2}, \quad T_d = \frac{m_d}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}I_d\dot{\varphi}_1^2 \tag{1.3}$$

$$U_r = \frac{k_r}{2}[(x_2 - x_1)^2 + (y_2 - y_1)^2], \quad T_r = \frac{m_r}{2}(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}I_r\dot{\varphi}_2^2 \tag{1.4}$$

where I_d and I_r are the central axial moments of inertia of the disc and the ring. Expressions (1.3) and (1.4) enable us to set up the Langrangian $L = T - U$.

We will write the generalized forces applied to the system under the different conditions of motion using the work of these forces

$$\delta A = \sum_{n=1}^{6} Q_n\delta q_n$$

On the other hand, the expression for the work can be written in the form

$$\delta A = \delta A_1 + \delta A_2; \quad \delta A_1 = \mathbf{f}_{fr1}\delta\mathbf{r}_{01}, \quad \delta A_2 = \int d\mathbf{f}_{fr2}\delta\mathbf{r}_{rel}$$

where $\mathbf{f}_{fr1}$ is the friction force which arises during the motion of the disc (the second term in expression (1.1)) and $\mathbf{f}_{fr2}$ is the friction force between the disc and the ring (the second term in (1.2)). After substituting into Langrange's equations

$$\frac{d}{dt}\frac{dL}{d\dot{q}_i} - \frac{dL}{dq_i} = Q_i$$

expressions for the Langrangian $L = T - U$, obtained using relations (1.3) and (1.4), and omitting the intermediate steps, we have

$$\begin{aligned}
&m_d\ddot{x}_1 + k_d x_1 + k_r(x_1 - x_2) + k_{2d}(x_1^2 + y_1^2)^{1/2}x_1 = \\
&= \eta_r l_r[(\dot{x}_2 - \dot{x}_1) + (y_2 - y_1)\dot{\varphi}_1] - \eta_d\dot{x}_1 - \eta_{2d}(\dot{x}_1^2 + \dot{y}_1^2)^{1/2}\dot{x}_1 \\
&m_d\ddot{y}_1 + k_d y_1 + k_r(y_1 - y_2) + k_{2d}(x_1^2 + y_1^2)^{1/2}y_1 = \\
&= \eta_r l_r[(\dot{y}_2 - \dot{y}_1) - (x_2 - x_1)\dot{\varphi}_1] - \eta_d\dot{y}_1 - \eta_{2d}(\dot{x}_1^2 + \dot{y}_1^2)^{1/2}\dot{y}_1 \\
&m_r\ddot{x}_2 + k_r(x_2 - x_1) = -\eta_r l_r[(\dot{x}_2 - \dot{x}_1) + (y_2 - y_1)\dot{\varphi}_1] \\
&m_r\ddot{y}_2 + k_r(y_2 - y_1) = -\eta_r l_r[(\dot{y}_2 - \dot{y}_1) + (x_2 - x_1)\dot{\varphi}_1] \\
&I_d\ddot{\varphi}_1 = -\eta_r l_r R_r^2(\dot{\varphi}_2 - \dot{\varphi}_1) + M_{dr} \\
&I_d\ddot{\varphi}_2 = -\eta_r l_r R_r^2(\dot{\varphi}_2 - \dot{\varphi}_1)
\end{aligned} \tag{1.5}$$

where $l_r = 2\pi R_r$ and M_{dr} is the moment created by the drive.

The system of equations (1.5) has the steady solution

$$x_1 = 0, \quad x_2 = 0, \quad y_1 = 0, \quad y_2 = 0, \quad \dot{\varphi}_1 = \Omega, \quad \dot{\varphi}_2 = \Omega \tag{1.6}$$

Linearizing system (1.5) in the neighbourhood of the steady solution (1.6) under the assumption that the angular velocity of rotation of the disc is maintained constant and equal to Ω by means of a special drive and introducing the complex variables

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2 \tag{1.7}$$

we obtain the system

$$\begin{aligned}
&m_d\ddot{z}_1 + k_d z_1 + k_r(z_1 - z_2) = \eta_r l_r[(\dot{z}_2 - \dot{z}_1) - i(z_2 - z_1)\Omega] - \eta_d\dot{z}_1 \\
&m_r\ddot{z}_2 + k_r(z_2 - z_1) = -\eta_r l_r[(\dot{z}_2 - \dot{z}_1) - i(z_2 - z_1)\Omega] \\
&I_r\ddot{\psi}_2 = -\eta_r l_r R_r^2\dot{\psi}_2 \quad (\dot{\psi}_2 = \dot{\varphi}_2 - \Omega)
\end{aligned} \tag{1.8}$$

The last of these equations, which has a solution of the form

$$\dot{\psi}_2 = C\exp(-\eta_r l_r R_r^2 I_r^{-1} t)$$

is separated out from the first two.

The equations of system (1.8) are invariant under a shift of the origin of the time reference and a rotation of the system of coordinates about the vertical axis by an angle of $\pi/2$. By virtue of these symmetry properties, the system admits of the particular solutions

$$z_1 = \hat{z}_1\exp(\lambda t), \quad z_2 = \hat{z}_2\exp(\lambda t) \tag{1.9}$$

The quantities $\hat{z}_1$, $\hat{z}_2$, λ satisfy equations which, after introducing the variables

$$w_1 = \hat{z}_1, \quad w_2 = \hat{z}_2 - \hat{z}_1$$

are simplified somewhat and reduce to the following

$$\begin{aligned} m_d w_1 \lambda^2 + k_d w_1 - k_r w_2 &= \eta_r l_r [w_2 \lambda - i w_2 \Omega] - \eta_d w_1 \lambda \\ m_r w_2 \lambda^2 + m_r w_1 \lambda^2 + k_r w_2 &= -\eta_r l_r [w_2 \lambda - i w_2 \Omega] \end{aligned} \tag{1.10}$$

The characteristic equation of the resulting system has the form

$$\begin{vmatrix} m_d \lambda^2 + \eta_d \lambda + k_d & -\eta_r l_r \lambda - k_r + i \eta_r l_r \Omega \\ m_r \lambda^2 & m_r \lambda^2 + \eta_r l_r \lambda + k_r - i \eta_r l_r \Omega \end{vmatrix} = 0 \tag{1.11}$$

Since λ depends continuously on the parameters of the problem, a change in the degree of instability of the system occurs when an imaginary number $\lambda = i\omega$ appears.

We will now determine the values of the damping factors η_d and the stiffness of the mounting k_d of the spindle of the disc which, when the remaining parameters are fixed, ensure imaginary values of λ. Substituting the expressions $\lambda = i\omega$ and $\eta_r = \sigma + i\chi$ into Eq. (1.11) and separating real and imaginary parts, we obtain a linear system of two equations in k_d and η_d. Solving these equations and introducing the dimensionless variables and parameters

$$K_d = \frac{k_d}{m_d \Omega^2}, \ H_d = \frac{\eta_d}{m_d \Omega}, \ \Pi_1 = \frac{\sigma l_r}{m_r \Omega}, \ \Pi_2 = \frac{\chi l_r}{m_r \Omega}, \ \tau = \frac{\omega}{\Omega}, \ \mu = \frac{m_d}{m_d + m_r}, \ K_r = \frac{k_r}{m_r \Omega^2} \tag{1.12}$$

we have

$$\begin{aligned} K_d &= \frac{\tau^2}{\mu \Delta} \{ [K_r - \Pi_2(\tau - 1) - \tau^2][K_r - \Pi_2(\tau - 1) - \mu \tau^2] + \Pi_1^2 (\tau - 1)^2 \} \\ H_d &= -\frac{\tau^3 (\tau - 1)(1 - \mu) \Pi_1}{\mu \Delta} \end{aligned} \tag{1.13}$$

where

$$\Delta = [\tau^2 + \Pi_2(\tau - 1) - K_r]^2 + \Pi_1^2 (\tau - 1)^2$$

Relations (1.13), in the case of fixed parameters Π_1, Π_2, μ, K_r and variable τ, defined a curve in the plane of the mounting parameters K_d and H_d which, together with the special line $K_d = 0$, form a so-called D-decomposition [6] of the above plane into domains with differing degrees of instability.

A D-decomposition (the solid curve with small dashes) of the $K_d H_d$ plane is shown in Fig. 3 for the following values: $\Pi_1 = 0.069$, $\Pi_2 = -4.2$, $\mu = 0.373$, $K_r = 1.204$.

The hatching of the boundaries of the D-decomposition is carried out in the following way: the passage across the boundary of a domain $D(n)$ from the unhatched side onto the hatched side corresponds to doubling of the order of instability, that is, a transfer into the domain $D(n + 2)$. The stability domain is denoted by $D(0)$. The direction in which the parameter τ increases is shown by an arrow. As the precession frequency tends to infinity ($\tau \to \pm\infty$), the mounting parameters $H_d \to -(\mu^{-1} - 1)\Pi_1$, $K_d \to +\infty$. Note that, in the case of the scale chosen in Fig. 3, the flattened branches of the D-curve (the upper and lower of which correspond to $-\infty < \tau < 0$ and $7.9 < \tau < \infty$) practically merge with the abscissa axis and the hatching of one branch is superimposed on the other. There are two stability domains in the case of the choice of physically interesting parameters. One stability domain, that is, $D_1(0)$, contains a point which corresponds to fairly large positive values of the damping factor H_d.

In order to demonstrate the effect of a change in the parameter Π_2 on the sizes of domains with a different degree of instability, the D-curves for $\Pi_2 = -3.5$ (the dashed curve) and $\Pi_2 = -4.8$ (the dot–dash curve) have been constructed in Fig. 3.

From the practical point of view, the stability domain $D_2(0)$, which is located in the neighbourhood of zero values of the parameters of the mounting of the spindle of the rotating disc, is of the greatest

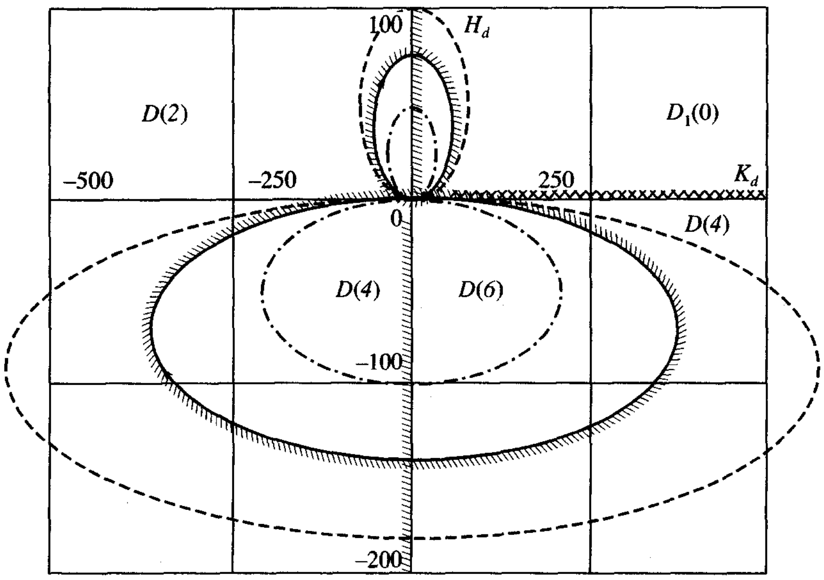

Fig. 3

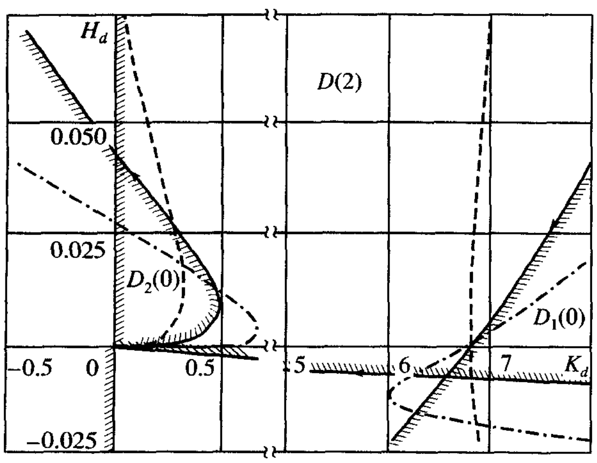

Fig. 4

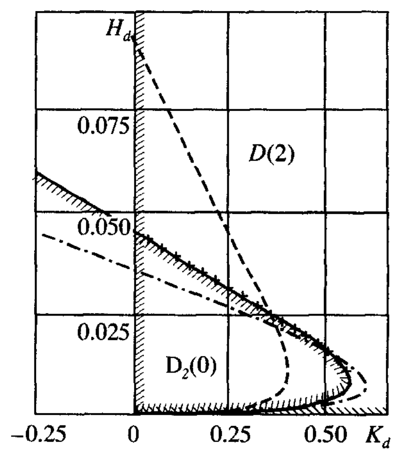

Fig. 5

interest (see Fig. 4, where a D-decomposition close to the origin of the system of coordinates is shown for $\Pi_1 = 0.028$, $\Pi_2 = -4.2$, $\mu = 0.373$, $K_r = 1.4$). D-decomposition curves are also shown in Fig. 4 for the cases $\Pi_2 = -3$ (the dashed curve) and $\Pi_2 = -6$ (the dot–dashed) with the remaining parameters unchanged. It can be seen from the graph that a reduction in Π_2 leads to an elongation of the domain $D_2(0)$ along the ordinate axis and, at the same time, it undergoes a small compression along the abscissa axis. The whole figure becomes smaller when Π_2 is increased.

The effect of a change in the parameter K_r when $\Pi_1 = 0.028$, $\Pi_2 = -4.2$, $\mu = 0.373$ is shown in Fig. 5. The dashed curve corresponds to the value $K_1 = 2.0$ and the dot–dash curve to the value $K_r = 0.95$. When K_r is increased, the stability domain $D_2(0)$ becomes larger along the ordinate axis and the domain $D_1(0)$ expands somewhat due to the approach of its boundary to the ordinate axis. Note that, as τ increases, the principal D-curve, after intersecting the ordinate axis when $K_r > 1$, is deflected upwards and forms a loop in the upper half plane but, when $K_r < 1$, it is deflected downwards, forming a loop which is predominantly in the lower half-plane.

2. A ROTOR PARTIALLY FILLED WITH A FLUID

The stability of a rotor is partially filled with a viscous, incompressible fluid, the angular velocity of which is kept constant by means of a special drive, was considered earlier in [2]. The cavity of the rotor has the shape of a tall cylinder. The effect of gravity is ignored in view of the high angular velocities of rotation. The spindle of the rotor is located in viscoelastic mountings, the angular movements of the spindle can be neglected and it can be assumed that all the particles of the rotor can only execute plane-parallel motions. In order to construct the D-decomposition, an approach is used in which no knowledge of the characteristic equation in explicit form is required.

After some reduction, the solution of the linearized problem of the plane-parallel motion of the fluid accompanying the angular precession of the rotor [2] can be represented in the form

$$u = \left[c_1 + \frac{c_2}{r^2} + i\frac{Z_1(kr)}{r}\right]e^{i\varphi} + \text{c.c.}$$

$$\upsilon = \left[ic_1 - \frac{ic_2}{r^2} - kZ_0(kr) + \frac{Z_1(kr)}{r}\right]e^{i\varphi} + \text{c.c.} \tag{2.1}$$

$$\frac{p}{\rho} = \left[i(2\Omega - \omega_0)c_1 r + i(2\Omega + \omega)\frac{c_2}{r} + \frac{\omega^2 r}{2} - 2\Omega Z_1(kr)\right]e^{i\varphi} + \text{c.c.}$$

where u and υ are the radial and azimuthal components of the velocity of motion of the fluid particles, p is the pressure, ρ is the density of the fluid, Ω is the absolute angular velocity of the rotor, ω is the angular velocity of precession, $Z_n(kr) = c_3H_n^{(2)}(kr) + c_4H_n^{(1)}(kr)$, $H_n^{(1),(2)}(kr)$ is an n-th order Hankel function, $k = \kappa(i - \omega_0/|\omega_0|)$, $\kappa = \sqrt{|\omega_0|/2\nu}$, $\omega_0 = \Omega - \omega$ is the angular velocity of proper rotation of the rotor and ν is the kinematic viscosity of the fluid.

We now change to the dimensionless variables

$$K_1 = \frac{k_1}{m_1\Omega^2},\quad H_1 = \frac{\eta_1}{m_1\Omega},\quad \tau = \frac{\omega}{\Omega},\quad \mu = \frac{m_1}{m_1 + m_2},\quad E = \frac{\nu}{\Omega a^2}$$

$$C_1 = \frac{c_1}{\Omega},\quad C_2 = \frac{c_2}{a^2\Omega},\quad C_3 = \frac{c_3H_0^{(2)}(ka)}{a\Omega},\quad C_4 = \frac{c_4H_0^{(1)}(kb)}{a\Omega}$$

where m_1 is the mass of the rotor, m_2 is the mass of the filling fluid, K_1 and H_1 are the dimensionless coefficient of elasticity and the dimensionless damping factor of the mountings of the rotor spindle respectively, a is the radius of the cylindrical cavity in the rotor, b is the internal radius of the cylindrical layer of fluid under conditions of steady rotation and E is the dimensionless Ekman number. The constants $C_1, \dots, C_4$ are determined from the boundary conditions on the free surface and on the inner wall of the rotor after substituting the solution of (2.1) into them

$$C_1 + C_2 + ih_{1a}C_3 + ig_{1a}C_4 = 0$$

$$C_1 - C_2 + i(ka - h_{1a})C_3 + i(kag_{0a} - g_{1a})C_4 = 0$$

$$4\delta^{-2}C_2 - i[2kbh_{0b} + (k^2b^2 - 4)\delta^{-1}h_{1b}]C_3 - i[2kb + (k^2b^2 - 4)\delta^{-1}g_{1b}]C_4 = 0 \tag{2.2}$$

$$-i\frac{\tau^2}{1-\tau}C_1 + i\delta^{-2}[1 - \tau - f(\tau)]C_2 +$$

$$+ \delta^{-1}\left[-\frac{2(1-\tau)}{kb}h_{0b} + f(\tau)h_{1b}\right]C_3 + \delta^{-1}\left[-\frac{2(1-\tau)}{kb} + f(\tau)g_{1b}\right]C_4 = -\frac{1}{2}\tau^2\Omega$$

where

$$h_{jb} = \frac{H_j^{(2)}(kb)}{H_0^{(2)}(ka)}, \quad g_{ja} = \frac{H_j^{(1)}(ka)}{H_0^1(kb)}, \quad j = 0, 1; \quad h_{1a} = \frac{H_1^{(2)}(ka)}{H_0^{(2)}(ka)}, \quad g_{1b} = \frac{H_1^{(1)}(kb)}{H_0^{(1)}(kb)}$$

$$\delta = \frac{b}{a}, \quad f(\tau) = \frac{2\tau - 1}{1 - \tau} + \frac{4(1 - \tau)}{k^2 b^2}$$

After integrating the stresses applied to the inner surface of the cylindrical cavity of the rotor, expressions are obtained for the components of the dimensionless hydrodynamic force acting per unit length of the cavity,

$$F_\xi = \frac{1}{1 - \delta^2}\mathrm{Re}\left(1 + 4i\frac{1 - \tau}{\tau^2}C_2\right), \quad F_\eta = -\frac{4}{1 - \delta}\frac{1 - \tau}{\tau^2}\mathrm{Re}\,C_2 \tag{2.3}$$

Substituting the expressions for the forces (2.3) into the equations for the transnational motion of the rotor, we obtain relations connecting the parameter τ and the defining parameters of the problem for the case of circular precession,

$$\frac{\mu}{1 - \mu}(-\tau^2 + K_1) = F_\xi \tau^2, \quad \frac{\mu}{1 - \mu}H_1 \tau = F_\eta \tau^2 \tag{2.4}$$

The graph of the hydrodynamic force as a function of the frequency of precession has a clearly expressed resonance from which is caused by resonance perturbation of the waves which propagate over the free surface of the fluid. The resonance frequencies can be approximately determined from the equality

$$\tau_{1,2} = (1 + \delta^2)^{-1}(2 \pm \sqrt{2(1 - \delta^2)}).$$

An example of a D-decomposition was given earlier in [2].

3. THE CHOICE OF THE PARAMETERS OF THE DISCRETE MODEL

The basic D-curve of the discrete model intersects the abscissa axis $H_d = 0$ at the origin of the coordinates $K_d = 0$ and at point A when $K_d = (\mu^{-1}K_r - 1)/(K_r - 1)$. We will find the points of intersection with the ordinate axis $K_d = 0$ when the parameter Π_1 is small and can be neglected. The following values of the parameter τ

$$\tau_{1,4} = \frac{1}{2\mu}(-\Pi_2 \mp \sqrt{\Pi_2^2 + 4\mu\Pi_2 + 4\mu K_r}), \quad \tau_{2,3} = -\frac{1}{2}\Pi_2 \mp \frac{1}{2}\sqrt{\Pi_2^2 + 4\Pi_2 + 4K_r}$$

correspond to the points of intersection. First of all, point B, which is closest to the abscissa axis, where the main curve intersects the ordinate axis $K_d = 0$ is of intersect since this point is, in fact, associated with the most intersecting stability domain adjoining the origin of the system of coordinates in the K_d, H_d plane. The ordinate of point B corresponds to the value $\tau = \tau_1$.

The main D-curve in the continuous model when $\mu = 0.37313$, $E = 10^{-6}$ and $\delta = 0.9$ intersects the abscissa axis at the point $K_1 = 11.664$ and the ordinate axis at the point $H_1 = 0.05289$ (the solid curve in Fig. 6, the dashed curve has not been plotted). Requiring that the points of intersection of the axes coincide and the closest positioning of the right-hand boundary of the domain $D_2(0)$ when $\mu = 0.37313$ in the discrete and continuous models, we obtain that, in the discrete model, it is necessary to take $K_r = 1.187$, $\Pi_1 = 0.77$, $\Pi_2 = -6.5$. With this choice of parameters, the parts of the D-curves of the continuous and discrete models, which are close to the origin of the system of coordinates form the right-hand boundary of the domain $D_2(0)$ in the K_1, H_1 plane, they pass close to one another and are indistinguishable in the scale of Fig. 5. At the same time, the figure as a whole, which are formed by the D-curves, do differ as is shown in Fig. 6 (the dotted curve corresponds to the discrete model). By a choice of the parameters of the discrete model, it is possible to achieve a significantly greater overall convergence of the two curves but, in this case, accuracy in the approximation of the boundary of the domain $D_2(0)$ is lost.

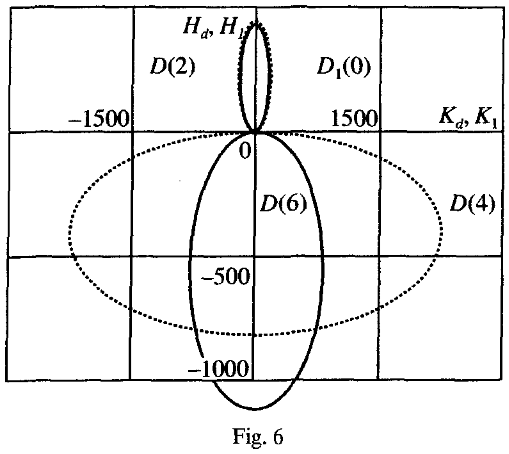

Fig. 6

4. "SAFE" AND "UNSAFE" BOUNDARIES OF THE STABILITY DOMAIN

An important question, concerning the behaviour of a rotor system in the case of parameter values which are close to the boundary of the stability domain (BSD), is associated with the creation from a state of equilibrium of a type of focal point of the limit cycle (or with the contraction of a cycle to an equilibrium state). We shall show that, in the finite-dimensional model of a rotor being considered, the angular velocity of which is kept constant and which has non-linear, viscoelastic spindle mountings, Andronov–Hopf bifurcation (see [4, 5, 7–10] occurs on leaving the stability domain.

After some simple reduction of the first four equations of system (1.5), we obtain

$$\begin{aligned} &m_d\ddot{z}_1 + k_d z_1 - k_r(z_2 - z_1) + k_{2d}|z_1|^{\alpha} z_1 = \\ &= \eta_r l_r[(\dot{z}_2 - \dot{z}_1) - i(z_2 - z_1)\Omega] - \eta_d \dot{z}_1 - \eta_{2d}|\dot{z}_1|^{\beta}\dot{z}_1 \\ &m_r\ddot{z}_2 + k_r(z_2 - z_1) = -\eta_r l_r[(\dot{z}_2 - \dot{z}_1) - i(z_2 - z_1)\Omega] \end{aligned} \tag{4.1}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. We shall seek the solution of system (4.1) in the form of a circular precession

$$z_1 = \varepsilon_1 \exp(i\omega t), \quad z_2 = (\varepsilon_1 + \varepsilon_2)\exp(i\omega t) \tag{4.2}$$

Substituting expressions (4.2) into system (4.1), we arrive at a system of non-linear equations in ε_1, ε_2 and ω which, after changing to dimensionless variables and parameters (see also (1.12))

$$\mathrm{E}_1 = \frac{\varepsilon_1}{R_d}, \quad \mathrm{E}_2 = \frac{\varepsilon_2}{R_d}, \quad K_{2d} = \frac{k_{2d}R_d^{\alpha}}{m_d\Omega^2}, \quad H_{2d} = \frac{\eta_{2d}R_r^{\beta}\Omega^{\beta-1}}{m_d}$$

can be written in the form

$$\begin{aligned} &(-\tau^2 + iH_d\tau + K_d)\mathrm{E}_1 + (\mu^{-1} - 1)\zeta(\tau)\mathrm{E}_2 = -K_{2d}|\mathrm{E}_1|^{\alpha}\mathrm{E}_1 - iH_{2d}|\mathrm{E}_1|^{\beta}|\tau|^{\beta}\tau\mathrm{E}_1 \\ &\tau^2\mathrm{E}_1 + [\tau^2 + \zeta(\tau)]\mathrm{E}_2 = 0; \quad \zeta(\tau) = (\tau - 1)(\Pi_2 - i\Pi_1) - K_r \end{aligned} \tag{4.3}$$

Note that the condition

$$\mathbf{AE} = 0 \tag{4.4}$$

where

$$\mathbf{A} = \begin{Vmatrix} -\tau^2 + iH_d\tau + K_d & (\mu^{-1} - 1)\zeta(\tau) \\ -\tau^2 & -\tau^2 - \zeta(\tau) \end{Vmatrix}, \quad \mathrm{E} = \begin{Vmatrix} \mathrm{E}_1 \\ \mathrm{E}_2 \end{Vmatrix}; \quad \det\mathbf{A} = 0$$

is satisfied on the BSD in the plane of the parameters of the mounting.

We will denote the deviations from the dimensionless parameters of elasticity K_d and viscosity H_d of the mountings and the frequency of precession τ corresponding to the BSD by δK_d, δH_d, $\delta\tau$. In the case when the deviations δK_d, δH_d are of small magnitude, after separating the real and imaginary parts from system (4.3), we obtain the following equations for $\delta\tau$ and $|E_1|$

$$K_{2d}|E_1|^{\alpha}-\frac{dK_d(\tau)}{d\tau}\delta\tau = -\delta K_d$$
$$H_{2d}|\tau E_1|^{\beta}-\frac{dH_d(\tau)}{d\tau}\delta\tau = -\delta H_d \tag{4.5}$$

For the case when $\alpha = \beta$, it follows from system (4.5) that

$$|E_1|^{\alpha} = \frac{l_2\delta K_d - l_1\delta H_d}{l_1 H_{2d}|\tau|^{\alpha}-l_2 K_{2d}}; \quad l_1 = \frac{dK_d(\tau)}{d\tau}, \quad l_2 = \frac{dH_d(\tau)}{d\tau} \tag{4.6}$$

where l_1 and l_2 are the components of the tangent vector $\mathbf{l} = (l_1, l_2)$ to BSD. The small increments δK_d and δH_d are chosen in such a way that the vector $\mathbf{n} = (\delta K_d, \delta H_d)$ is perpendicular to the BSD. If it is stipulated that the increments δK_d and δH_d must be chosen such that the vector $\mathbf{n}$ is directed into the stability domain $D_2(0)$, formula (4.6) can be rewritten in the form

$$|E_1|^{\alpha} = l_2^{-1}\left(K_{2d}-\frac{dK_d}{dH_d}H_{2d}|\tau|^{\alpha}\right)^{-1}|[\mathbf{l}, \mathbf{n}]| \tag{4.7}$$

where $|[\mathbf{l}, \mathbf{n}]|$ is the modulus of the vector product. Since $l_2 > 0$ on the BSD $D_2(0)$, the sign of the right-hand side of equality (4.7) is determined by the expression in parenthesis.

If a periodic motion is produced on leaving the stability domain across any segment of its boundary, then such a bifurcation is called a supercritical bifurcation. The generation of self-excited oscillations on passing across such a segment occurs softly and such segments of the BSD have been called "safe". In Fig. 5, the "unsafe" segment are indicated by crosses. In the neighbourhood of "unsafe" segments, a periodic motion in the form of a circular precession of small radius exists in the domain of stability of a state of steady rotation. This means that, on approaching such a segment from the stability domain, the system becomes unstable to perturbations of small, but finite, magnitude. It can be seen from formula (4.7) that, theoretically, any segment of the BSD can be made "safe" by choosing the magnitude of K_{2d} and H_{2d} (if, of course, materials with suitable non-linear properties can be found). For instance, when $K_{2d} < 0$ (that is, in the case of so-called soft or regressing elastic non-linearity) and $H_{2d} = 0$, the whole of the BSD $D_2(0)$ is "safe". Note that $dK_d/dH_d > 0$ on the lower part of the BSD up to the furthest right-hand point of the graph (we denote it by C) and $dK_d/dH_d < 0$ on the part of the BSD located above the point C. The positive values of H_{2d}, in the case soft elastic non-linearity, have no effect on the character of the lower part of the BSD (up to the point C) and the upper part can be partially or completely converted into "unsafe" boundary for certain values of H_{2d}. This case is shown in Fig. 5. When $\mu = 0.373$, $K_r = 1.204$, $\Pi_1 = 0.069$, $\Pi_2 = -4.2$ and $\alpha = 1$, an "unsafe" segment appears in the upper part of the boundary when $K_{2d} > -18H_{2d}$.

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